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The conditions for a discontinuous function to be identical with the value function of a game in a time-optimal problem☆

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Abstract

A differential game is considered in which the time until a point reaches a target set is the pay functional. The sufficient conditions for the given discontinuous function to be identical with the value function of the game are obtained. The conditions are formulated in terms of the classical concepts of u- and v-stable functions but it is additionally required that the so-called condition of the proper contractibility of the closed sets of the level of the function being tested is satisfied. An example is presented which shows that the condition of proper contractibility is not excessive.

This paper is concerned with the problem of finding the conditions that are imposed on a given discontinuous function which, when satisfied, are sufficient to match it to the value function of the time-optimal differential game being considered. The investigations are carried out within the framework of the positional formalization of differential games introduced by Krasovskii and Subbotin.[1]

In the theory of differential games, control problems are studied using the principle of feedback under conditions where there is uncertainty and interference. A useful control is considered as an action of the first player which minimizes a certain functional in the set of trajectories of the system and the interference is assumed to be the result of a control action of the second player whose aim is to maximize the same functional. The classical approach[1–4] to the solution of a differential game involves searching for a value function which, at each point of the space of the states of the system, matches an optimal guaranteed result in a game starting from this point. Optimal control strategies using the principle of feedback are constructed on the basis of the value function.

In the case of a differentiable value function, the search for it reduces[2] to solving a boundary-value problem for a first-order partial differential equation (the Isaacs–Bellman equation).

If the value function is not smooth but continuous, then the concepts of continuous u-and v-stable functions (Ref. 3, p. 145), which were introduced into the theory of positional differential games, become fundamental in its characterization. In this case, u-stable (v-stable) functions with a corresponding boundary condition majorize (minorize) the value function of a differential game which is unique and possesses the properties of u- and v-stability.

The need to consider discontinuous value functions arises, for example, in time-optimal problems that leads to the concepts of semicontinuous u- and v-stable functions. The characterization of the value function is considerably more complicated in this case. That is, in time-optimal problems, the value function is a unique u-stable function which

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is semicontinuous from below and satisfies a null boundary condition on the boundary of the terminal set to which the sequence of υ-stable functions, which are semicontinuous from above and satisfy the same boundary condition, converges pointwise. Verification of the existence of such a sequence and, especially, its construction, are difficult even in the case of the solution of problems in a plane.

In this paper, sufficient conditions are proposed for a discontinuous test function to be identical with the value function of the time-optimal differential game being investigated. The conditions assume the semicontinuity from below and u-stability of the test function, the ν-stability of the upper closure of the test function and, also, that the condition introduced here of the *proper contractibility* of the closed sets of the level of the test function is satisfied.

The properties of u- and ν-stability have been thoroughly studied in the theory of differential games. Different infinitesimal criteria of the u- and ν-stability of semicontinuous functions have been obtained (Ref. 4, p. 38). We also note that the concept of a u-stable (υ-stable) function corresponds to the concept of an upper (lower) generalized viscous solution of a first-order partial differential equation (see Ref. 5, for example). Verification of the condition for the proper contractibility of a closed set is an independent problem. The solution of this problem is not obvious in the general case and is not investigated here.

In optimal control theory, the method of dynamic programming serves as an analogue of the approach to the solution of the problem on the basis of a value function. If the function of the optimal result (the Bellman function) is differentiable, then the search for it reduces to solving the corresponding boundary-value problem for a first-order partial differential equation. In this case, the optimal control is determined according to the feedback principle using the Bellman function. If the Bellman function is not smooth, but continuous, then a regular Boltyanskii synthesis (Ref. 6, p. 263) can be used to slove the problem in a class of controls using the feedback principle. In the case of a discontinuous Bellman function, the construction of the optimal control using the feedback principle and its substantiation rests, as a rule, on the special features of the dynamics of each specific problem.

The sufficient conditions for optimality proposed below also hold for control problems, since problems in control theory can be considered as a special case of problems in the theory of differential games (with a null constraint on the control of the second player). However, there are no simplifications whatsoever in the formulation of the conditions, that is, verification of the ν-stability of the upper closure of the test functions is required in spite of the fact that there is no second player.

1. Formulation of the problem

Consider a control system, the motion of which is described by the equation

$$\dot{x}(t) = f(x(t), u(t), \upsilon(t)), \quad t \geq 0. \tag{1.1}$$

Here $x(t) \in R^n$ is the phase state at the instant of time t, $u(t) \in P$ and $\nu(t) \in Q$ are the controls of the first (minimizing) and second (maximizing) players, and P and Q are compact sets. It is assumed that the function $f(x, u, \nu)$ is continuous with respect to the set of variables, it satisfies the inequality

$$\|f(x, u, \upsilon)\| \leq \kappa(1 + \|x\|), \quad \kappa = \text{const} > 0$$

and that the Lipschitz condition with respect to the variable x, that is,

$$\|f(x^{(1)}, u, \upsilon) - f(x^{(2)}, u, \upsilon)\| \leq \lambda(X)\|x^{(1)} - x^{(2)}\|, \quad \lambda(X) = \text{const} > 0$$

for all $x^{(1)}, x^{(2)} \in X$, $u \in P$, $\nu \in Q$, holds for each bounded domain $X \subset R^n$. Moreover, suppose the saddle point condition[1]

$$H(x, p) = \min_{u \in P} \max_{\upsilon \in Q} \langle p, f(x, u, \upsilon)\rangle = \max_{\upsilon \in Q} \min_{u \in P} \langle p, f(x, u, \upsilon)\rangle.$$

is satisfied for any $x, p \in R^n$. Here, a scalar product of vectors in denoted by the angular brackets.

The aim of the first player is the fastest convergence to a phase point $x(t)$ with a given closed set $M \subset R^n$. The second player strives either to avoid an encounter with M or to maximize the time until the encounter occurs.

The positional formalization of a time-optimal game[1] is used.

In the case of the above-mentioned conditions which are imposed on the function $f(x, u, \nu)$, a game value $T(x_0; M)$ exists[1] for any $x_0 \in R^n$. The function $T(\cdot; M)$: $R^n \to [0, \infty]$ is called the value function of the game.

The problem consists of finding conditions, which are imposed on the function $\varphi(\cdot)$: $\Omega \to [0, \infty]$, for which the equality $\varphi(x) = T(x; M)$, $x \in \Omega$ is satisfied. Here $\Omega \subseteq R^n$ is a closed set and $M \subset \Omega$. The required conditions must only utilize the properties of the function $\varphi(\cdot)$ and not require any additional constructions.

2. The properties of the value function of a game

We will now present the basic properties of the value function which we shall subsequently use.

We introduce the notation

$$W(t; M) = \{x \in R^n : T(x; M) \le t\}, \quad t \ge 0.$$

The equality

$$T(x; M) = T(x; W(\tau; M)) + \tau.$$

is satisfied for any $\tau > 0$ and any $x \notin W(\tau; M)$.

It follows from well-known results[1,4] that $T(\cdot; M)$ is a function which is semicontinuous from below, $M = W(0; M)$ and that the following property of u-stability is satisfied.

Property (T_u). For any $y_0 \in R^n \backslash M$, $v_* \in Q$ and $\tau > 0$, a solution $y(\cdot)$: $[0, \tau] \to R^n$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(y(t), u, v_*), u \in P\}, \quad y(0) = y_0,$$

exists such that either the inequality $T(y(\tau); M) \le T(y_0; M) - \tau$ is satisfied or $y(t) \in M$ for a certain $t \in [0, \tau]$. The symbol "co" denotes the convex hull of a set.

We note that it immediately follows from the property T_u that, if $x_* \in R^n \backslash M$ and $T(x_*; M) < \infty$, then a sequence $\{x_n\}_1^\infty$ exists such that $T(x_n; M) < T(x_*; M)$ and $x_n \to x_*$ when $n \to \infty$.

In the case of an upper closure

$$T^*(x; M) = \limsup_{y \to x} T(y; M)$$

of the value function, the following property of v-stability is satisfied (Ref. 4, p. 258).

Property (T_v). For any $y_0 \in R^n \backslash M$, $u_* \in P$ and $\tau > 0$, a solution $y(\cdot)$: $[0, \tau] \to R^n$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(y(t), u_*, v), v \in Q\}, \quad y(0) = y_0,$$

exists such that the inequality

$$T^*(y(\tau); M) \ge T^*(y_0; M) - \tau.$$

is satisfied.

3. Properly contractible sets

Suppose $\mathbf{M} \subset R^n$ is a closed set and $\mathrm{int}\mathbf{M}$ is the interior of the set $\mathbf{M}$. Subject to the condition that $\mathrm{int}\mathbf{M} \neq \emptyset$, we put

$$\mathbf{M}^{[\varepsilon]} = \{x \in \mathbf{M} : \mathbf{B}(x, \varepsilon) \subseteq \mathbf{M}\}, \quad \varepsilon > 0; \quad \varepsilon_{\mathbf{M}} = \max\{\varepsilon > 0 : \mathbf{M}^{[\varepsilon]} \neq \emptyset\}$$

where $\mathbf{B}(x, \varepsilon)$ is a closed sphere of radius ε with its centre at the point x.

Lemma 3.1. Suppose $\mathbf{M}$ is a closed set, $\mathrm{int}\mathbf{M} \neq \emptyset$, $x_* \in R^n$ and

$$\lim_{\varepsilon \to +0} T(x_*; \mathbf{M}^{[\varepsilon]}) = T(x_*; \mathbf{M}). \tag{3.1}$$

The function $T(\cdot; \mathbf{M})$ is then continuous at the point x_*.

Proof. Suppose $\vartheta_* = T(x_*; \mathbf{M})$. We choose an arbitrary sequence $\{x_n\}_1^\infty$ which converges to x_*. It is required to prove that

$$\lim_{n\to\infty} T(x_n; \mathbf{M}) = \vartheta_*. \tag{3.2}$$

Taking account of the semicontinuity from below of the function $T(\cdot; \mathbf{M})$, we have

$$\lim_{n\to\infty} T(x_n; \mathbf{M}) \geq \vartheta_*. \tag{3.3}$$

We find from this that, if $\vartheta_* = \infty$, equality (3.2) is satisfied.

Moreover, if $x_* \in \mathrm{int}\mathbf{M}$, equality (3.2) also holds.

We will now assume that $\vartheta_* < \infty$ and $x_* \notin \mathrm{int}\mathbf{M}$. Suppose $\tau > 0$. By virtue of relation (3.1), an $\varepsilon > 0$ is found such that $T(x_*; \mathbf{M}^{[\varepsilon]}) \leq \vartheta_* + \tau$. Consequently, a set $W_u \subset [0, \vartheta_* + \tau] \times R^n$ exists[1] which possesses the property of u-stability with respect to the set $[0, \vartheta_* + \tau] \times \mathbf{M}^{[\varepsilon]}$ and $(0, x_*) \in W_u$.

If the first player uses a strategy of extremal aiming into the set W_u from the initial point x_n, and a measurable programmed control $v(\cdot)$ with values in the set Q is realized for the second player, then the estimate (Ref. 1, p.62)

$$\min_{t\in[0,\vartheta_*+\tau]} \mathrm{dist}(x(t), \mathbf{M}^{[\varepsilon]}) \leq \|x_* - x_n\| \exp(\lambda(X)(\vartheta_* + \tau)).$$

holds for the constructive motion $x(\cdot)$ of system (1.1) which arises. Here, $\lambda(X)$ if the Lipschitz constant with respect to x of the function f in a certain bounded domain $X \subset R^n$ which contains all the motions being considered, and $\mathrm{dist}(x(t), \mathbf{M}^{[\varepsilon]})$ is the distance from the point $x(t)$ to the set $\mathbf{M}^{[\varepsilon]}$.

Consequently, if

$$\|x_* - x_n\| \leq \varepsilon \exp(-\lambda(X)(\vartheta_* + \tau)),$$

then

$$\min_{t\in[0,\vartheta_*+\tau]} \mathrm{dist}(x(t), \mathbf{M}) \leq 0 \text{ и } T(x_n; \mathbf{M}) \leq \vartheta_* + \tau.$$

Taking account of inequality (3.3), we have

$$\vartheta_* = T(x_*; \mathbf{M}) \leq \liminf_{n\to\infty} T(x_n; \mathbf{M}) \leq \limsup_{n\to\infty} T(x_n; \mathbf{M}) \leq \vartheta_* + \tau.$$

Since the magnitude of τ was chosen arbitrarily, then, on taking the limit as $\tau \to 0$, we obtain inequality (3.2).

Definition 3.1. A set $\mathbf{M} \subset R^n$ is said to be *properly contractible* with respect to the dynamics (1.1) if a number $\vartheta > 0$ exists such that the following conditions are satisfied.

Condition C1. $W(\vartheta; \mathbf{M}) \neq \mathbf{M}$ and $W(t; \mathbf{M}) = \mathrm{int}W(t; \mathbf{M})$, $t \in [0, \vartheta]$.
Condition C2. The equality

$$\lim_{\varepsilon\to+0} T(x; \mathbf{M}^{[\varepsilon]}) = T(x; \mathbf{M}).$$

is satisfied for any point $x \in \mathrm{int}W(\vartheta; \mathbf{M})\backslash\mathbf{M}$.

Note that, if $W(\vartheta; \mathbf{M}) \neq \mathbf{M}$, then, by virtue of the T_u property of u-stability, the function $T(\cdot; \mathbf{M})$ must take all values from the interval $(0, \vartheta)$. From this we obtain that, for a properly contractible set $\mathbf{M}$, the quantity $\vartheta > 0$, which satisfies Conditions C1 and C2, can be chosen to be as small as desired.

We will now present the simplest cases of the proper contractibility of a set $\mathbf{M}$.

Suppose a set $\mathbf{M}$ has a smooth boundary $\partial\mathbf{M}$, $\mathbf{M} = \overline{\mathrm{int}\mathbf{M}}$ and that, for any point $x \in \partial\mathbf{M}$, the inequality

$$\min_{u\in P}\max_{v\in Q} \langle \nu(x), f(x, u, v)\rangle < 0, \tag{3.4}$$

is satisfied, where $\nu(x)$ is the outward normal to the set $\mathbf{M}$ at a point $x \in \partial\mathbf{M}$.

We select $\vartheta > 0$ and $x_* \in W(\vartheta; \mathbf{M})$. It follows from condition (3.4) and the definition of the value of a game that, for any $\tau > 0$, the first player guarantees getting from the point x_* into the set int$\mathbf{M}$ in a time interval $[0, T(x_*; \mathbf{M}) + \tau]$. Consequently, an $\varepsilon_* > 0$ is found such that

$$T(x_*; \mathbf{M}^{[\varepsilon_*]}) \le T(x_*; \mathbf{M}) + \tau.$$

Whence, on taking account that there is no increase in the values of $T(x_*; \mathbf{M}^{[\varepsilon]})$ when $\varepsilon \to +0$, we obtain the limit relation

$$\lim_{\varepsilon \to +0} T(x_*; \mathbf{M}^{[\varepsilon]}) = T(x_*; \mathbf{M}).$$

Hence, from the definition of the proper contractibility of a set condition C2 is satisfied for any $\vartheta > 0$.

For any $x_* \in R^n \backslash \cup_{\vartheta \ge 0} W(\vartheta; \mathbf{M})$, we have

$$T(x_*; \mathbf{M}) = T(x_*; \mathbf{M}^{[\varepsilon_*]}) = \infty, \quad \varepsilon \in (0, \varepsilon_{\mathbf{M}}].$$

and, by virtue of Lemma 3.1, we obtain the continuity of the value function $T(\cdot; \mathbf{M})$ in R^n.

Note that the conditions for the continuity of the value function under weaker assumptions with respect to the set $\mathbf{M}$ have been proved earlier in Ref. 5.

The correctness of Condition C1 for any $\vartheta > 0$ follows from the continuity of the function $T(\cdot; \mathbf{M})$ and the property T_u.

Hence, the set $\mathbf{M}$ is properly contractible. More complex sufficient conditions for the proper contractibility of sets are associated with a discontinuous value function and they are not considered here.

4. Formulation of a theorem on the sufficient conditions

We will now formulate a basic assertion.

Theorem. *Suppose $\Omega \subseteq R^n$ and $M \subset \Omega$ are closed sets, the function $\varphi(\cdot)$: $\Omega \to [0, \infty]$ is semicontinuous from below and the following conditions are satisfied.*

Condition A1. The equality $\varphi(x) = 0$, $x \in M$ holds.

Condition A2 (u-stability). For any $y_0 \in \Omega \backslash M$, $v_* \in Q$ and $\tau > 0$ a solution $y(\cdot)$: $[0, \tau] \to \Omega$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(y(t), u, v_*), u \in P\}, \quad y(0) = y_0,$$

exists such that either the inequality

$$\varphi(y(\tau)) \le \varphi(y_0) - \tau,$$

is satisfied or $y(t) \in M$ for a certain $t \in [0, \tau]$.

Condition A3 (v-stability). For any $y_0 \in \Omega \backslash M$, $u_* \in P$ and $\tau > 0$ a solution $y(\cdot)$: $[0, \tau] \to R^n$ of the differential inclusion

$$\dot{y}(t) \in \mathrm{co}\{f(y(t), u_*, v), v \in Q\}, \quad y(0) = y_0,$$

exists such that the inequality

$$\varphi^*(y(\tau)) \ge \varphi^*(y_0) - \tau,$$

is satisfied where

$$\varphi^*(x) = \begin{cases} \limsup\limits_{z \to x} \varphi(z), & x \in \mathrm{int}\Omega \\ \sup\limits_{z \in \Omega} \varphi(z), & x \notin \mathrm{int}\Omega. \end{cases} \tag{4.1}$$

Condition A4. The sets of the level

$$D(t) = \{x \in \Omega : \varphi(x) \le t\}, \quad 0 < t < \sup_{z \in \Omega} \varphi(z)$$

are properly contractible.

Then,

$$\varphi(x) = T(x; M), \quad x \in \Omega. \tag{4.2}$$

We shall call the function $\varphi^*(\cdot)$: $R^n \to [0, \infty]$, which is semicontinuous from above and defined by formula (4.1), the upper closure of the function $\varphi(\cdot)$: $\Omega \to [0, \infty]$.

Remark 1. Suppose Conditions A1–A3 are satisfied for the function $\varphi(\cdot)$ and the condition for the proper contractibility of the sets $D(t)$ is only violated at a certain single point $a \in (0, \sup \varphi(z))$. Then, the theorem on sufficient conditions can be applied from the beginning to the function $\varphi(\cdot)$: $D(a) \to [0, \infty]$. This gives the equality $\varphi(x) = T(x; M)$, $x \in D(a)$. Then, on introducing the notation

$$M_1 = D(a), \quad \varphi_1(x) = \begin{cases} \varphi(x) - a, & x \notin M_1 \\ 0, & x \in M_1 \end{cases}$$

the theorem can be applied to the function $\varphi_1(\cdot)$: $\Omega \to [0, \infty]$ and to a differential game with a terminal set M_1. From this, using the relation $T(x; M_1) = T(x; M) - a$, we obtain the equality $\varphi(x) = T(x; M)$ for all $x \in \Omega$.

Similarly, the theorem can be applied in the case when the proper contractibility of the sets of a level $D(t)$ of the function $\varphi(\cdot)$ is violated at a finite number of points from the interval $(0, \sup_{z \in \Omega} \varphi(z))$.

Remark 2. It follows from the T_u and T_v properties of the value function that Conditions A1–A3 are necessary for the value function of a game. If the function $\varphi(\cdot)$ is continuous, Conditions A1–A3 will be necessary and sufficient[4] to satisfy equality (4.2).

In Section 6, we will give an example which shows that, in the case of a continuous function $\varphi(\cdot)$, Conditions A1–A3 are insufficient to satisfy equality (4.2), that is, Condition A4 cannot be discarded.

5. Proof of the theorem on sufficient conditions

We shall first prove some auxiliary assertions.

Lemma 5.1. Suppose a closed set $M \subset R^n$ and a function $\varphi_1(\cdot)$: $\Omega \to [0, \infty]$, which is semicontinuous from below, satisfy Conditions A1 and A2. Then,

$$T(x_0; M) \le \varphi(x_0), \quad x_0 \in \Omega.$$

Proof. If $x_0 \in M$, then, according to Condition A1, we have

$$\varphi(x_0) = T(x_0; M) = 0.$$

If $x_0 \in \Omega \backslash M$ and $\varphi(x_0) = \infty$, then $T(x_0; M) \le \infty = \varphi(x_0)$.

Suppose $x_0 \in \Omega \backslash M$, $\vartheta_* = \varphi(x_0) < \infty$. We will now show that

$$T(x_0; M) \le \vartheta_*. \tag{5.1}$$

We consider the set

$$W_* = \left\{(t, x) \in [0, \vartheta_*] \times R^n \backslash M : \varphi(x) \le \vartheta_* - t\right\}.$$

It follows from Condition A2 that W_* is a u-stable bridge[1] in the problem of the adduction of a point $(t, x(t))$ into the set $[0, \vartheta_*] \times M$. Since, $(0, x_0) \in W_*$, the first player possesses[1] a positional strategy which guarantees getting into the set M in a time interval $[0. \vartheta_*]$.

From the definition of the value of a game, for any $\delta > 0$ the second player possesses a positional strategy which guarantees evasion of the neighbourhood of the set M in a time interval $[0, T(x_0; M) - \delta]$ for any actions of the first player. From this, we have the inequality $T(x_0; M) - \delta < \vartheta_*$ for any $\delta > 0$. Consequently, inequality (5.1) holds.

Lemma 5.2. Suppose a closed set $M \subset R^n$ and the function $\varphi(\cdot)\colon \Omega \to [0, \infty]$, which is semicontinuous from below, satisfy Conditions A1 and A3. Moreover, suppose $\mathrm{int} M \neq \emptyset$. Then,

$$\varphi^*(x_0) \le T(x_0; M^{[\varepsilon]}), \quad x_0 \in R^n, \quad \varepsilon \in (0, \varepsilon_M].$$

Proof. Suppose $x_0 \in R^n$ and $\vartheta^* = \varphi^*(x_0) < \infty$. We will assume that

$$W^* = \{(t, x) \in [0, \vartheta^*] \times R^n : \varphi^*(x) \ge \vartheta^* - t\}.$$

It follows from Condition A3 that W^* is a v-stable set.[1] Since $(0, x_0) \in W^*$, the second player possesses[1] a positional strategy which guarantees that the system is retained in the set W^* in the time interval $[0, \vartheta^*]$. Since

$$([0, \vartheta^*] \times M^{[\varepsilon]}) \cap W^* = \emptyset, \quad \varepsilon \in (0, \varepsilon_M],$$

the second player evades a certain neighbourhood of the set $M^{[\varepsilon]}$ in the time interval $[0, \vartheta^*]$ for any actions of the first player. However, the first player possesses a positional strategy which guarantees getting into $M^{[\varepsilon]}$ in the time interval $[0, T(x_0; M^{[\varepsilon]})]$. Hence

$$\vartheta^* < T(x_0; M^{[\varepsilon]}), \quad \varepsilon \in (0, \varepsilon_M].$$

Suppose $x_0 \in R^n$ and $\varphi^*(x_0) = \infty$. We will assume that

$$W_\infty = [0, \infty) \times \{x \in R^n : \varphi^*(x) = \infty\}.$$

The set W_∞ is closed and $(0, x_0) \in W_\infty$. It follows from Condition A3 that W_∞ is a v-stable set.Since $\varphi^*(x) = 0$ for any $x \in \mathrm{int} M$, we have

$$([0, \infty) \times \mathrm{int} M) \cap W_\infty = \emptyset.$$

Consequently,

$$([0, \infty) \times M^{[\varepsilon]}) \cap W_\infty = \emptyset, \quad \varepsilon \in (0, \varepsilon_M].$$

Hence, for any $\vartheta > 0$, we obtain that the second player possesses[1] a positional strategy which guarantees evasion of the neighbourhood of the set $M^{[\varepsilon]}$ in the time interval $[0, \vartheta]$. This means that $T(x_0; M^{[\varepsilon]}) = \infty$ for all $\varepsilon \in (0, \varepsilon_M)$.

Lemma 5.3. Suppose the closed sets $D_\tau \subset R^n$, $\tau > 0$ decrease monotonically along the inclusion when $\tau \to +0$ and $\cap_{\tau>0} D_\tau = M$. Then,

$$\lim_{\tau \to +0} T(x; D_\tau) = T(x; M), \quad x \in R^n. \tag{5.2}$$

Proof.

1°. A closed ε-neighbourhood of the terminal set M is denoted by M_ε, that is,

$$M_\varepsilon = \{z_1 + z_2 : z_1 \in M, \|z_2\| \le \varepsilon\}.$$

We will show that

$$T(x; M) = \lim_{\varepsilon \to +0} T(x; M_\varepsilon), \quad x \in R^n. \tag{5.3}$$

The equality (5.3) is obvious for all $x \in M$. We select $x \in R^n \setminus M$.

Suppose $T(x; M) < \infty$. Since $M \subset M_\varepsilon$, then

$$T(x; M_\varepsilon) \le T(x; M). \tag{5.4}$$

It follows from well-known results[1] that, for any $\delta > 0$, a $\varepsilon = \varepsilon(\delta) > 0$ exists such that the second player evades the set M_ε from the point x in the time interval $[0, T(x; M) - \delta]$. Consequently,

$$T(x; M) - \delta < T(x; M_{\varepsilon(\delta)}). \tag{5.5}$$

Taking account of inequalities (5.4) and (5.5), we obtain equality (5.3) for the point x.

Suppose $T(x; M) = \infty$. According to well-known results,[1] for any $K > 0$ an $\varepsilon > 0$ exists such that the second player evades the set M_ε in the time interval $[0,K]$, that is, $T(x, M_\varepsilon) > K$. Consequently,

$$\lim_{\varepsilon \to +0} T(x; M_\varepsilon) = \infty$$

and equality (5.3) is satisfied for the point x.

2°. Suppose $x_* \in R^n$ and $\vartheta = T(x_*; M) < \infty$. A closed bounded set $Z \subset R^n$ exists which contains all possible trajectories of system (1.1) with the initial condition $x(0) = x_*$ in the time interval $[0, \vartheta]$.

For a number $\tau > 0$, we define the value

$$\varepsilon(\tau) = \min\{\varepsilon : D_\tau \cap Z \subseteq M_\varepsilon \cap Z\}.$$

We have $\varepsilon(\tau) \to +0$ when $\tau \to +0$.

The inequalities

$$T(x_*; M_{\varepsilon(\tau)}) \le T(x_*; D_\tau) \le T(x_*; M).$$

follow from the imbeddings $M \subseteq D_\tau u D_\tau \cap Z \subseteq D_\tau \subseteq M_{\varepsilon(\tau)}$, $\tau > 0$.

From this, when account is taken of relation (5.3), we obtain

$$\lim_{\tau \to +0} T(x_*; M_{\varepsilon(\tau)}) = \lim_{\tau \to +0} T(x_*; D_\tau) = T(x_*; M).$$

Hence, equality (5.2) is proved.

3°. Suppose $x_* \in R^n$ and $T(x_*; M) = \infty$. By virtue of relation (5.3), we have

$$\lim_{\varepsilon \to +0} T(x_*; M_\varepsilon) = \infty,$$

We select an arbitrary $K > 0$ and denote the closed bounded set in the space R^n, which contains all of the trajectories of system (1.1) with the initial condition $x(0) = x_*$ in the time interval $[0,K]$, by $Z(K)$. An $\varepsilon = \varepsilon(K) > 0$ is found such that $T(x_*; M_\varepsilon) > K$. Since $\cap_{\tau > 0} D_\tau = M$, a $\tau = \tau(K) > 0$ exists such that $D_\tau \cap Z(K) \subset M_\varepsilon \cap Z(K)$. Consequently, $T(x_*; D_\tau) > K$. Since the number K was chosen arbitrarily, we obtain the equality

$$\lim_{\tau \to +0} T(x_*; D_\tau) = \infty,$$

and relation (5.2) is satisfied.

Proof of the theorem. If $M = \Omega$, then the conclusion of the theorem is obvious.

Suppose $M \neq \Omega$. Condition A1 gives the relation $W(0; M) = M \subseteq D(0)$. It follows from Condition A2 of u-stability that there are no points of a local minimum of the function $\varphi(\cdot)$ outside the set M at which it takes finite values. Hence, we obtain the equality $W(0; M) = D(0)$.

Since $M \neq \Omega$ and $D(0) = M$, then $\sup\limits_{x \in \Omega} \varphi(x) > 0$.

We choose $\tau \in (0, \sup\limits_{x \in \Omega} \varphi(x))$ and define the function $\varphi_\tau(\cdot)$: $\Omega \to [0, \infty]$ as follows:

$$\varphi_{\tau(x)} = \begin{cases} \varphi(x) - \tau, & x \notin D(\tau) \\ 0, & x \in D(\tau). \end{cases}$$

We will now show that

$$T(x; D(\tau)) = \varphi_\tau(x), \quad x \in \Omega. \tag{5.6}$$

For brevity, we will put $D_\tau = D(\tau)$. The set D_τ and the function $\varphi_\tau(\cdot)$, which is semicontinuous from below, satisfy Conditions A1–A4 in which the notation M and $\varphi(\cdot)$ is replaced by D_τ and $\varphi_\tau(\cdot)$.

Suppose

$$\Theta = \sup_{x \in \Omega} \varphi_\tau(x), \quad E(t) = \{x \in \Omega : \varphi_\tau(x) \le t\}$$

$$\gamma = \sup\{\vartheta \in [0, \Theta) : W(t; D_\tau) = E(t)\ \forall t \in [0, \vartheta]\}.$$

Note that the equality $\varphi_\tau(x) = T(x; D_\tau)$ is satisfied for any $\vartheta \in [0, \gamma]$ and $x \in E(\vartheta)$. In fact, suppose $t = T(x; D_\tau)$. Since $E(\vartheta) = W(\vartheta; D_\tau)$, we have $t \in [0, \vartheta]$ and $x \in W(t; D_\tau) = E(t)$. The inequality $t \le \varphi_\tau(x)$ is satisfied by virtue of Lemma 5.1. On the other hand, since $x \in E(\mathrm{t})$, we have $\varphi_\tau(x) \le t$. Consequently, $\varphi_\tau(x) = T(x; D_\tau)$.

We will now consider the following cases.

Case 1: $\gamma = \infty$. For any $\vartheta \ge 0$ and $x \in E(\vartheta)$, we have $\varphi_\tau(x) = T(x; D_\tau)$. If $x \in \Omega \backslash \cup_{\vartheta \ge 0} E(\vartheta)$ then, on taking account of the definition of the quantity γ, we find that $x \notin \cup_{\vartheta \ge 0} W(\vartheta; D_\tau)$ and, consequently, $T(x; D_\tau) = \infty$. By virtue of Lemma 5.1, we obtain the equality $\varphi_\tau(x) = \infty$. Hence, relation (5.6) has been proved.

Case 2: $\gamma < \infty$ and $\gamma = \Theta$. If $x \in E(\vartheta)$ for a certain $\vartheta \in [0, \gamma]$, equality (5.6) is satisfied.

Suppose $x \in \Omega \backslash \cup_{\vartheta \in [0, \gamma]} E(\vartheta)$. Taking account of Lemma 5.1, we have

$$\varphi_\tau(x) \ge T(x; D_\tau) \ge \gamma = \Theta.$$

By virtue of the definition of the number Θ, we obtain the equality $T(x; D_\tau) = \gamma$. The existence of a sequence $\{x_k\}_1^\infty$ such that $t_k = T(x_k; D_\tau) < \gamma$ and $x_k \to x$ when $k \to \infty$ follows from the property T_u of the u-stability of the value function. Since $W(t_k; D_\tau) = E(t_k)$ and $x_k \in E(t_k)$, the equality $\varphi_\tau(x_k) = T(x_k; D_\tau)$ holds.

By virtue of Lemma 5.1 and the function $\varphi_\tau(\cdot)$, which is semicontinuous from below, we obtain

$$\gamma = T(x; D_\tau) \le \varphi_\tau(x) \le \limsup_{k \to \infty} \varphi_\tau(x_k) = \limsup_{k \to \infty} T(x_k; D_\tau) = \limsup_{k \to \infty} t_k \le \gamma.$$

Hence, relation (5.6) is proved.

Case 3: $\gamma \in [0, \Theta]$. We introduce the notation

$$\mathbf{M} = E(\gamma), \quad \tilde{\varphi}(x) = \begin{cases} \varphi_\tau(x) - \gamma, & x \notin \mathbf{M} \\ 0, & x \in \mathbf{M}, \end{cases} \qquad \mathbf{D}(t) = \{x \in \Omega : \tilde{\varphi}(x) \le t\}.$$

We have $\sup\limits_{x \in \Omega} \tilde{\varphi}(x) = \Theta - \gamma$.

The set $\mathbf{M}$ and the function $\tilde{\varphi}(\cdot) : \Omega \to [0, \infty]$, which is semicontinuous from below, satisfy Conditions A1–A3 in which the notation M and $\varphi(\cdot)$ has been replaced by $\mathbf{M}$ and $\tilde{\varphi}(\cdot)$.

Taking account of condition A4 of the theorem and the equality $\mathbf{M} = D(\tau + \gamma)$, where $0 < \tau + \gamma < \sup\limits_{x \in \Omega} \varphi(x)$, we find that the set $\mathbf{M}$ is properly contractible. Consequently, an $\vartheta > 0$ exists such that Conditions C1 and C2 are satisfied. We choose the number ϑ such that $\vartheta < \Omega - \gamma$.

We will now show that

$$W(t; \mathbf{M}) = \mathbf{D}(t), \quad t \in (0, \vartheta). \tag{5.7}$$

We choose $t \in (0, \vartheta)$. Taking account of Lemma 5.1, we obtain

$$\mathbf{D}(t) \subseteq W(t; \mathbf{M}) = \operatorname{int} W(t; \mathbf{M}).$$

We will assume that

$$W(t; \mathbf{M}) \backslash \mathbf{D}(t) \neq \varnothing. \tag{5.8}$$

Since $\varphi(\cdot)$ is a function which is semicontinuous from below, then $\mathbf{D}(t)$ is a closed set. Suppose $x \in \operatorname{int} W(t; \mathbf{M}) \backslash \mathbf{D}(t)$. We have $T(x; \mathbf{M}) \in (0, t)$. If $x \in \Omega$, then the equality $t < \tilde{\varphi}(x)$ is obtained from the relation $x \notin \mathbf{D}(t)$. Hence, $t < \tilde{\varphi}^*(x)$, where $\tilde{\varphi}^*(\cdot)$ is the upper closure of the function $\tilde{\varphi}(\cdot)$ which is defined by formula (4.1). If $x \notin \Omega$, then $\varphi^*(x) = \Theta - \gamma > t$. Consequently,

$$T(x; \mathbf{M}) \le t < \tilde{\varphi}^*(x). \tag{5.9}$$

On the other hand, the inequality

$$\tilde{\varphi}^*(x) \le T(x; \mathbf{M}^{[\varepsilon]}), \quad \varepsilon \in (0, \varepsilon_{\mathbf{M}}].$$

is satisfied on the basis of Lemma 5.2. From Condition C2 for the proper contractibility of the set $\mathbf{M}$, we obtain that

$$T(x; \mathbf{M}) = \lim_{\varepsilon \to +0} T(x; \mathbf{M}^{[\varepsilon]}).$$

Consequently, $\tilde{\varphi}^*(x) \le T(x; \mathbf{M})$, which contradicts inequality (5.9). Hence, assumption (5.8) is untrue and property (5.7) is proved.

Taking account of the relation

$$W(t; \mathbf{M}) = W(\gamma + t; D_\tau), \quad \mathbf{D}(t) = E(\gamma + t), \quad t \in (0, \vartheta)$$

we obtain the equality $W(t; D_\tau) = E(t)$ for any $t \in [0, \gamma + \vartheta]$, which contradicts the definition of γ. Consequently, the case when $\gamma \in [0, \Theta]$ is impossible.

Hence, relation (5.6) is proved.

Suppose $x_* \in \Omega \backslash M$. Then, for small $\tau > 0$, we have $\varphi_\tau(x_*) = \varphi(x_*) - \tau$. By virtue of Lemma 5.3 and equality (5.6), we obtain

$$T(x_*; M) = \lim_{\tau \to +0} T(x_*; D(\tau)) = \lim_{\tau \to +0} \varphi_\tau(x_*) = \varphi(x_*)$$

whence the assertion of the theorem follows. □

6. Example

We will now present an example which shows that Condition A4 cannot be discarded in the case of a continuous function $\varphi(\cdot)$.

Suppose the dynamics of the system have the form

$$\dot{x} = Ax + u + \upsilon, \quad x \in R^2, \quad u \in P, \quad \upsilon \in Q, \tag{6.1}$$

where

$$A = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}$$

$$P = \{u \in R^2 : u_1 = 0, |u_2| \le 1\}, \quad Q = \{\upsilon \in R^2 : |\upsilon_1| + |\upsilon_2| \le 1\}$$

(A is a rotation matrix).

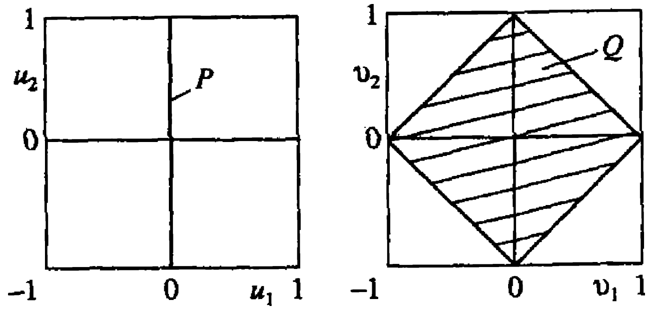

Fig. 1.

The constraints P and Q imposed on the controls of the players are shown in Fig. 1. The set m is specified using the system of inequalities (Fig. 2)

$$x_1 \geq 2, \quad x_2 \geq 1, \quad \rho^+(x) \leq 3\sqrt{2}, \quad \rho^-(x) \geq \sqrt{2}; \quad \rho^{\pm}(x) = \sqrt{(x_1 \pm 1)^2 + (x_2 \pm 1)^2}.$$

If the control actions of the players are constant, that is, $u(t) = u^*$, $v(t) = v^*$, the motion $x(t)$ of system (6.1) is a clockwise rotation around the point x^c which is a zero of the right-hand side of system (6.1). Since $A^{-1} = -A$, then

$$x^c = A(u^* + v^*). \tag{6.2}$$

1°. Suppose $m_1 = (2, 2)^T$, $m_2 = (-2, 2)^T$ and l is the smaller arc of a circle of radius $2\sqrt{2}$ with its centre at the origin of the coordinates joining the points m_1 and m_2. We will now show that

$$T(x; M) = \infty, \quad x \in R^2 \backslash (l \cup M). \tag{6.3}$$

We put

$$Z_1 = \{x \in R^2 : x_2 < x_1, x_1 > 0, x_2 \geq 0\}$$

$$Z_2 = \{x \in R^2 : x_2 < -x_1, x_1 < 0, x_2 \geq 0\}$$

$$l_1 = \{x \in Z_1 : \rho^+(x) = 3\sqrt{2}\}, \quad l_2 = \{x \in Z_2 : (x_1 - 1)^2 + (x_2 + 1)^2 = (3\sqrt{2})^2\}.$$

We denote the part of the set Z_i consisting of points strictly lying below the arc l_i by Z_i^- $(i = 1, 2)$. Suppose

$$Z_i^+ = Z_i \backslash Z_i^-, \quad Z = \{x \in R^2 : x_2 \geq |x_1|\}$$

Z^- (Z^+) is the part of the set Z lying strictly below (above) the arc l. The sets $Z^{\pm}$, $Z_1^{\pm}$, $Z_2^{\pm}$ are shown in Fig. 2.

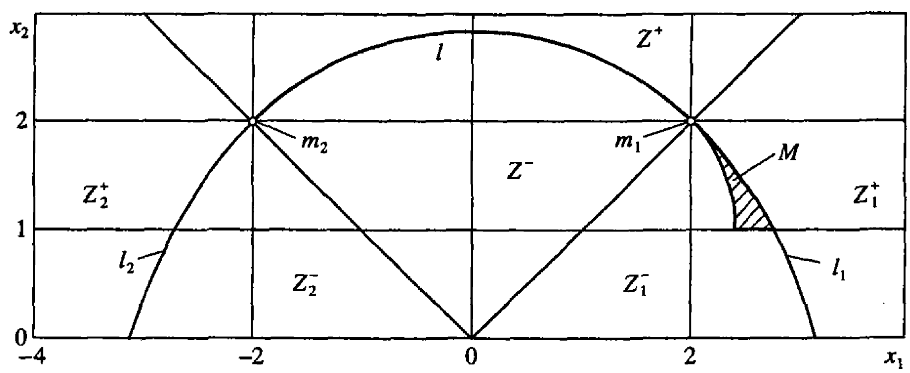

Fig. 2.

In the set $R^2\backslash(l\cup M)$, we now define the strategy of the second player in the form

$$V^0(x) = \begin{cases} (0, \pm 1)^T, & x \in Z^{\pm} \\ (\pm 1, 0)^T, & x \in Z_1^{\pm} \cup Z_2^{\mp} \\ \upsilon \in Q, & x_2 < 0. \end{cases}$$

We will show that the strategy $V^0(x)$ does not allow the trajectory of the system to get into the set $l\cup M$ from any initial point $x_0 \in R^2\backslash(l\cup M)$.

Suppose $x_0 \in Z^+$. Then, the second player applies a constant control $\upsilon^* = (0, 1)^T$ prior to emerging from the set Z^+. If the first player uses a control $u^* \in P$, then we obtain the following conditions, which the coordinates of the centre of rotation

$$0 \le x_1^c \le 2, \quad x_2^c = 0.$$

must satisfy, from equality (6.2). Consequently, for any action of the first player, the trajectory of system (6.1) will not lie below an arc of the circle of radius $||x_0||$ with centre at the origin of coordinates and, on emerging from Z^+, it will intersect the line $x_1 = x_2$ above the point m_1. Similarly, in the case of the set Z^-, we have

$$-2 \le x_1^c \le 0, \quad x_2^c = 0.$$

Hence, the trajectory of system (6.1) will not lie above an arc of a circle of radius $||x_0||$ with its centre at the origin of coordinates and, on emerging from Z^-, will turn out to be on the line $x_1 = |x_2|$ below the points m_1 and m_2.

Suppose $x_0 \in Z_1^+ \cup Z_2^-$. Then, the second player applies a constant control $v^* = (1, 0)^T$ prior to emerging from the set $Z_1^+ \cup Z_2^-$. From equality (6.2), we obtain the conditions

$$-1 \le x_1^c \le 1, \quad x_2^c = -1.$$

Consequently, on emerging from Z_1^+, the trajectory of system (6.1) intersects the line $x_2 = 0$ strictly above the arc l_1 and, on emerging from Z_2^-, it intersects the lines $x_2 = -x_1$ and $x_2 = 0$ strictly below the arc l_2 for any actions of the first player.

Suppose $x_0 \in (Z_1^- \cup Z_2^+)\backslash M$. Then, on emerging from the set $Z_1^- \cup Z_2^+$, the second player uses the constant control $v^* = (-1, 0)^T$. From equality (6.2), we obtain the conditions

$$-1 \le x_1^c \le 1, \quad x_2^c = 1.$$

Consequently, on emerging from Z_2^+, the trajectory of the system intersects the line $x_2 = -x_1$ above the point m_2 and, during its motion in Z_1^-, it does not intersect the set M and either gets onto the line $x_2 = 0$ which is strictly below the arc l_1 or onto the line $x_2 = x_1$ which is strictly below the point m_1. Hence, equality (6.3) is proved.

2°. We will now find the significance $T(x; M)$ of the value of the game for any $x \in l$.

Suppose the system is at a point $z = z(t) \in l$ at a certain instant of time t. For any $\bar{\upsilon} = \bar{\upsilon}(t) \in Q$, we define $\bar{u} = \bar{u}(t, \bar{\upsilon}) \in P$ such that the velocity vector is directed along a tangent to the arc l. Since the tangent to the arc l at the point z is perpendicular to the vector z, then

$$z_1(z_2 + \bar{\upsilon}_1) + z_2(-z_1 + \bar{\upsilon}_2 + \bar{u}_2) = 0.$$

Expressing the value of $\bar{u}_2$ from here and taking account of the constraint $\bar{u} \in P$, we obtain

$$\bar{u}(t, \bar{\upsilon}) = (0, -z_1\bar{\upsilon}_1/z_2 - \bar{\upsilon}_2)^T.$$

The slowest motion along the arc l will be in the case of a choice by the second player of a control $\bar{\upsilon} = \upsilon^*(t)$ from the condition for minimizing the quantity

$$\|Az(t) + \bar{u}(t, \bar{\upsilon}) + \bar{\upsilon}\|^2 = (z_2 + \bar{\upsilon}_1)^2 + (z_1 + z_1\bar{\upsilon}_1/z_2)^2$$

with respect to all $\bar{\upsilon} \in Q$. We have $\upsilon^*(t) = (-1, 0)^T$.

Suppose $z(0) = m_2$ and that, at the instant of time t, the players use the control actions $\upsilon^*(t)$ and $u^*(t) = \bar{u}(t, \upsilon^*(t))$. Then, the trajectory $z(t)$ of system (6.1) goes along the arc l and is described by the equations

$$\dot{z}_1 = z_2 - 1, \quad \dot{z}_2 = -z_1 + z_1/z_2.$$

The time when the trajectory $z(t)$ reaches the point m_1 is finite. In fact, since $z_2(t) \geq 2$, $z_1 \in [-2, 2]$, then $\dot{z}_1 = z_2 - 1 \geq 1$ and the instant ϑ_*, defined by the condition $z(\vartheta_*) = m_1$, is estimated by the inequality

$$\vartheta_* = \int_0^{\vartheta_*} dt \leq \int_{-2}^{2} dz_1 = 4.$$

For any $x \in l$, we denote an instant of time t which satisfies the equality $z(t) = x$ by $\tau(x)$. Hence, $\tau(x)$ is the greatest time for reaching the point x from the point m_2 in the case of motion along l and discrimination of the second player. We have $\tau(m_1) = \vartheta_*$, $\tau(m_2) = 0$. Taking account of the result obtained in Section 1°, we conclude that the set

$$W_* = \{(t, x) : t \in [0, \tau(x)], x \in l \cup M\}$$

is the maximum u-stable bridge in a convergence problem with a set $[0, \vartheta_*] \times M$. We have $(\tau(x), x) \in W_*$, $x \in l$.

Using the strategy of extremal aiming at the set W_*, the first player guarantees reaching the set M from the point m_2 in a time interval $[0, \vartheta_*]$. In the case of optimal control of the second player, the time taken to reach the set M is equal to ϑ_*. In the case of an arbitrary point $x_0 \in l$, we find that

$$T(x_0; M) = \vartheta_* - \tau(x_0).$$

3°. Suppose

$$\varphi(x) = 2T(x; M), \quad x \in R^2.$$

It follows from the property of u-stability of the function $T(\cdot; M)$ that the function $\varphi(\cdot)$ also possesses the property of u-stability. Since $\varphi^*(x) = T^*(x; M)$, $x \in R^2$, the function $\varphi^*(\cdot)$ possesses the property of u-stability. Here, $T^*(\cdot; M)$, $\varphi^*(\cdot)$ are the upper closures of the functions $T(\cdot; M)$, $\varphi(\cdot)$, which are defined by formula (4.1), where $\Omega = R^2$.

Hence, Conditions A1–A3 are satisfied in the case of the function $\varphi(\cdot) : R^2 \to [0, \infty]$ but $\varphi(x) \neq T(x; M)$, $x \in l \backslash \{m_1\}$.

The function $\varphi(\cdot)$ does not satisfy Condition A4 since Condition C1 is violated for all $\vartheta > 0$. Furthermore, since Condition A4 is also not satisfied in the case of the value function $T(\cdot; M)$, it is not a necessary condition.

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